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A note on the spin-up of a stratified fluid

By JAMES R. HOLTON
University of Washington
AND PETER H. STONE
Pierce Hall, Harvard University

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It is pointed out that recent theoretical results for the spin-up of a continuously stratified fluid with insulating boundary conditions are at variance with experiment. However, it is then shown that the theoretical analysis contains an inconsistent scaling assumption, so that the apparent contradiction is not a real one. The nature of the spin-up as shown by the experiments is briefly described.

In a recent paper, Pedlosky (1967) investigated theoretically the transient motion which results when a rotating cylinder containing a continuously and stably stratified fluid has its rotation rate impulsively changed by a small amount. He concluded that when the fluid is subject to insulating boundary conditions the adjustment to the new angular velocity of the vessel occurs exclusively on a diffusive time scale rather than the 'spin-up' time scale found by Greenspan & Howard (1963) for a homogeneous fluid. Pedlosky's result has been quoted by Dicke (1967) in support of Dicke's suggestion that the interior of the sun may be rotating much faster than its surface layers.

However, Pedlosky's conclusion is at variance with the theoretical and experimental results of Holton (1965). Holton performed spin-up experiments in a cylindrical tank using salt stratified water as the working fluid. He found that, just as in the homogeneous case, there are three distinct time scales in the adjustment process. (i) In time Ω_0^{-1} (where Ω_0 is the angular velocity of the vessel) Ekman boundary layers develop as a result of the viscous stresses on the horizontal boundaries. (ii) The secondary meridional circulation induced by the convergence in the Ekman layers spins the fluid up to a quasi-steady state in a time of order $H/(\Omega_0 \nu)^{\frac{1}{2}}$ (where H is the depth of the fluid and ν is the kinematic viscosity). However, unlike the homogeneous case, this quasi-steady state is not one of solid body rotation. It is a state in which the relative angular velocity is zero at the edge of the Ekman layers (hence there is no Ekman suction) but increases exponentially away from the Ekman layers with an e-folding distance inversely proportional to the static stability. This vertical shear of the relative angular velocity is geostrophically balanced by horizontal density gradients. (iii) Finally, in a time of order H^2/ν viscous diffusion brings the whole system to a state of solid body rotation.

In Holton's theoretical treatment of the second (spin-up) time scale he assumed that the secondary flow was driven solely by Ekman suction and that the side-wall boundary layers played a purely passive role in closing the circulation. The spin-up process in the interior differs from the homogeneous case because stratification suppresses the vertical motion away from the boundaries. Thus the angular momentum preserving meriodional inflow in the interior necessary to balance the outflow in the Ekman layer decreases exponentially away from the boundary.

Pedlosky (1967) attempted to obtain a solution for the side-wall boundary layer which would close the secondary circulation thereby confirming Holton's assumption. However, in the case of insulating boundaries (i.e. no flux of density across the boundary), using scaling arguments, he was not able to match an interior solution to a side-wall boundary layer. This result led him to assume that dissipative processes, rather than Ekman layer suction, were important in the fluid interior. Using this assumption, he found a solution to the equations, from which he concluded that '*...there are no Ekman layers and no side-wall boundary layers; the interior spins up by a strictly diffusive process...*' (his italics).

This apparent contradiction between theory and experiment is, however, illusory, for Pedlosky's solution is not consistent with his scaling assumptions. His solution assumes that the azimuthal velocity is in geostrophic equilibrium everywhere so that it is completely determined by the zero-order pressure field.† Consequently the initial conditions and boundary conditions (2.7) and (6.8) require initially a step function distribution of $p^{(0)}$ at $z = 0, 1$. Since $p^{(0)}$ is in hydrostatic balance, (6.35) implies an initial delta function distribution of perturbation density concentrated at $z = 0, 1$. In fact, by substituting the solution (6.37) into (6.35) it is easily shown that for $t \lesssim O(E)$, $\rho^{(0)} \gtrsim O(E^{-\frac{1}{2}})$ in boundary layers of depth $\lesssim O(E^{\frac{1}{2}})$ at $z = 0, 1$. Thus Pedlosky's solution violates his scaling assumption that $\rho^{(0)} \sim O(1)$ everywhere.

In addition, substitution of (6.37) into the density diffusion equation (6.33) with the aid of (6.35) indicates that for $t \lesssim O(E)$, $w \gtrsim O(E^{\frac{1}{2}})$. (Note that the $O(E^{-\frac{1}{2}})$ terms in this equation sum to zero.) Hence, by continuity $U \gtrsim O(1)$ in boundary layers of depth $\lesssim O(E^{\frac{1}{2}})$ at $z = 0, 1$. Thus the solution also violates the scaling assumption that $U \sim O(E)$, and indicates that the vertical frictional stress term cannot be neglected in the meridional momentum equation in regions of depth $\sim O(E^{\frac{1}{2}})$ near $z = 0, 1$. Consequently a correct theoretical analysis of the problem is still needed.

The fact that Pedlosky's solution is inconsistent with his scalings when $t \sim O(E)$ suggests that the initial development of the flow should be analysed

† His solution (equation 6.37) contains several misprints. It should read as follows:

$$p^{(0)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \left\{ \exp - \left[\frac{(n^2\pi^2 + \gamma_m^2)(4\delta S n^2\pi^2 + \gamma_m^2)}{(4Sn^2\pi^2 + \gamma_m^2)} \right] t \right\} \sin(n\pi z)\, J_0(\gamma_m r),$$

where

$$A_{mn} = \frac{8(1-(-1)^n)}{n\pi\gamma_m^2 J_0(\gamma_m)} \quad \text{and} \quad J_0'(\gamma_m) = 0.$$

All notation and equation numbers in this discussion are the same as in Pedlosky's paper.

by assuming this scale for t. This scale corresponds to the same period, Ω_0^{-1}, which Greenspan & Howard (1963) found for the setting up of Ekman layers in the homogeneous case. This result indicates that a correct theoretical analysis of the stratified spin-up problem would show a time development more like that for the homogeneous case, as was in fact found in Holton's experiments.

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